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Statistical accuracy in estimating parameters of the spatial coherence function by photon counting techniques

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Abstract. We investigate the statistical errors which arise in estimating parameters of the spatial coherence of gaussian optical fields by using two photon counters and a digital correlator. An expression for the error is analytically derived under the assumption that the counting time of incremental samples is much smaller than the light coherence time. Two examples are given: estimation of the spatial coherence length and estimation of the radius of a far incoherent object.

1. Introduction

Among the measurable quantities that describe an optical field, the intensity correlation function has been realized to contain valuable information about the light spectral shape as well as the dimensions and radiance distribution of its source (Hanbury Brown and Twiss 1956, 1957). With the development of photon counting techniques and digital correlation devices it has become possible to measure this function more accurately. The method of optical spectroscopy by digital autocorrelations has found an increasing number of applications and its theory is now well developed (Jakeman 1970). Moreover, the statistical accuracy in estimating parameters of the spectral linewidth has been calculated (Jakeman *et al* 1971a).

On the other hand, the use of photocorrelators for estimating spatial parameters of optical fields has not found the same recent attention (the effect of spatial correlation on the performance of photon correlators that estimate temporal parameters has been studied by Jakeman *et al* 1971b and Degiorgio *et al* 1971). We report here results of calculations of errors encountered in estimating parameters of the spatial coherence function of optical fields. The approach is similar to that of Jakeman *et al* (1971a) although in our case two detectors are used instead of one. The results are valid whenever the following assumptions apply. (i) The light field is gaussian, stationary and cross-spectrally pure (see eg Mandel and Wolf 1965). (ii) The counting time interval of each sample is much smaller than the field's coherence time. (iii) The total counting time, or duration of the experiment, is much larger than the coherence time of the optical field.

2. The optical field

Assume a stationary and cross-spectrally pure optical field whose complex degree of coherence at points r_1 and r_2 is

$$\gamma_{12} = \Gamma_{12}\chi(t_1 - t_2). \tag{1}$$

It is convenient to choose the normalization $\chi(0) = 1$, and to define the effective coherence length

$$\tau_c = \int_{-\infty}^{+\infty} |\chi(t)|^2 \, \mathrm{d}t, \tag{2}$$

and the bandwidth $W = \tau_c^{-1}$. It will be also necessary to use the higher order normalized intensity correlation functions $g_{12\ldots M}(t_1, t_2, \ldots, t_M)$. With the usual assumption of a complex circular gaussian optical field, these functions can be expanded in terms of γ_{12}. The relations to be used in this work are

$$g_{12} = 1 + |\gamma_{12}|^2 \tag{3a}$$

$$g_{123} = \sum_{\substack{i \neq j \neq k \\ = 1,2,3}} \gamma_{1i}\gamma_{2j}\gamma_{3k} \tag{3b}$$

$$g_{1234} = \sum_{\substack{i \neq j \neq k \neq l \\ = 1,2,3,4}} \gamma_{1i}\gamma_{2j}\gamma_{3k}\gamma_{4l}. \tag{3c}$$

3. The photodetectors

It is assumed that the optical field is detected using two point photodetectors at locations r_1 and r_2. The effect of the finite area of the detectors is not considered here. Let $n_j(t)$ be the number of photons counted in a time interval T_d centred around t by the detector located at r_j. We will be interested here in the statistical moments of $n_j(t)$. Expressions of these moments are in general difficult (see eg Jakeman 1970), however with our assumption that $\tau_c \gg T_d$, the statistical moments are related to the intensity correlation functions of the detected field by the simple equations

$$E(n_1(t_1)n_2(t_2)) = \bar{n}_1\bar{n}_2 g_{12}(t_1, t_2) \tag{4a}$$

$$E(n_1(t_1)n_1(t_2)n_2(t_3)) = \bar{n}_1^2\bar{n}_2 g_{112}(t_1, t_2, t_3) + \delta_{t_1,t_2}\bar{n}_1\bar{n}_2 g_{12}(t_1, t_3) \tag{4b}$$

$$\begin{aligned} E(n_1(t_1)&n_1(t_2)n_2(t_3)n_2(t_4)) \\ &= \bar{n}_1^2\bar{n}_2^2 g_{1122}(t_1, t_2, t_3, t_4) + \delta_{t_1,t_2}\bar{n}_1^2 n_2 g_{122}(t_1, t_3, t_4) \\ &\quad + \delta_{t_3,t_4}\bar{n}_1\bar{n}_2^2 g_{112}(t_1, t_2, t_3) + \delta_{t_1,t_2}\delta_{t_3,t_4}\bar{n}_1\bar{n}_2 g_{12}(t_1, t_4) \end{aligned} \tag{4c}$$

where δ is the Kronecker delta. Equations (4*a*, *b*, *c*) are valid only when $r_1 \neq r_2$. If $r_1 = r_2$ additional terms appear.

4. The digital correlator

The available observation time T is divided into N equal intervals each of duration T_d, and the number of counts in each interval is recorded. Let $n_j(m)$ be the number of counts in the mth interval by detector j. The digital correlator calculates the statistic

$$\hat{g}_{12}(l) = \frac{\hat{G}_{12}(l)}{\hat{n}_1\hat{n}_2} \tag{5a}$$

where

$$\hat{G}_{12}(l) = \frac{1}{N}\sum_{m=1}^{N} n_1(m)n_2(m+l) \tag{5b}$$

and

$$\hat{n}_j = \frac{1}{N}\sum_{m=1}^{N} n_j(m) \qquad j = 1, 2. \tag{5c}$$

This number can be used to estimate the normalized intensity correlation function $g_{12}(0, \tau_l)$, where $\tau_l = (l+\frac{1}{2})T_d$. In this work we aim at studying the use of the statistic $\hat{g}_{12}(l)$ in estimating parameters of the spatial coherence of the field. For this purpose, it is necessary to study the statistical properties of $\hat{g}_{12}(l)$. For simplicity in notations we write $\hat{g}_{12}(l)$ and $\hat{G}_{12}(l)$ simply as $\hat{g}$ and $\hat{G}$, and also $g_{12}(0, \tau_l)$ as g.

4.1. The mean

By expanding $\hat{n}_j$ around its average value $\bar{n}_j$, we can show that if $|(\hat{n}_j - \bar{n}_j)/\bar{n}_j| \ll 1$, for $j = 1, 2$, then to a first approximation

$$E(\hat{g}) \simeq g + \sum_{j=1,2}\left(\frac{1}{\bar{n}_j\bar{n}_1\bar{n}_2}E\{\hat{G}(\bar{n}_j - \hat{n}_j)\} + \frac{g}{\bar{n}_j^2}\operatorname{var}(\hat{n}_j)\right) + \frac{g}{\bar{n}_1\bar{n}_2}\operatorname{cov}(\hat{n}_1, \hat{n}_2). \tag{6}$$

By using the definitions ($5b$, c) and substituting from ($4a$, b) and ($3a$, b) it can be shown that the bias term $\{E(\hat{g}) - g\}$ is proportional to $(TW)^{-1}$ and thus is a very small number. It will be shown later that $(\operatorname{var}(\hat{g}))^{1/2}$ is of the order of $(TW)^{-1/2}$. Hence the main source of error is the uncertainties expressed by the variance and not the error due to the bias. We conclude that, to the first approximation, $\hat{g}$ is an unbiased estimator of g.

4.2. The variance

By expanding $\hat{n}_j$ around $\bar{n}_j$ and $\hat{G}$ around its average G we can show that

$$\operatorname{var}(\hat{g}) = \frac{\operatorname{var}(\hat{G})}{\bar{n}_1^2\bar{n}_2^2} + 4g^2 + \sum_{j=1,2}\left(-\frac{2g}{\bar{n}_1\bar{n}_2}\frac{E(\hat{G}\hat{n}_j)}{\bar{n}_j} + \frac{g^2}{\bar{n}_j^2}\operatorname{var}(\hat{n}_j)\right) + \frac{2}{\bar{n}_1\bar{n}_2}\operatorname{cov}(\hat{n}_1, \hat{n}_2). \tag{7}$$

In (7) we substitute $\hat{G}$ and $\hat{n}$ as given by ($5b$) and ($5c$) and thus write var ($\hat{g}$) in terms of the various moments of $n_j(m)$. We further use ($4a$, b, c) to write var (g) in terms of the intensity correlation function and then use the gaussian expansion ($3a$, b, c) to relate it to $\chi(\tau)$ and Γ_{12}. Making use of the assumption that $T_d \ll \tau_c$, the summations of ($5b$, c) can be written approximately as integrals. Also the assumption that $T \gg \tau_c$ simplifies the form of these integrals. We finally find the following expression:

$$\begin{aligned}\operatorname{var}(\hat{g}_{12}(\tau)) = (TW)^{-1}\Big\{&2 + a(0) + u(\tau_l)\Gamma_{12}^2 + v(\tau_l)\Gamma_{12}^4 + \left(\frac{1}{\bar{n}_{c1}} + \frac{1}{\bar{n}_{c2}}\right)(1 + 2\chi^2(\tau_l)\Gamma_{12}^2 \\ &- \chi^4(\tau_l)\Gamma_{12}^4) + \frac{1}{\bar{n}_{c1}\bar{n}_2}(1 + \chi^2(\tau_l)\Gamma_{12}^2) - \delta_{1,2}\frac{2}{\bar{n}_{c1}}(1 + \chi^2(\tau_l)\Gamma_{12}^2)^2\Big\},\end{aligned} \tag{8}$$

where $\bar{n}_{cj} = \bar{n}_j(\tau_c/T_d)$ is the average number of counts in a coherence time τ_c, and

$$a(\tau) = W \int_{-\infty}^{+\infty} |\chi(t)|^2 |\chi(t-\tau)|^2 \, dt \tag{9a}$$

$$u(\tau) = -2 + 6\chi^2(\tau) + 2W \operatorname{Re} \int_{-\infty}^{+\infty} \chi^*(t)\chi^*(t)\chi(t+\tau)\chi(t-\tau) \, dt \tag{9b}$$

$$v(\tau) = a(2\tau) - 4\chi^2(\tau) + 4\chi^4(\tau) - 8\chi^2(\tau) \operatorname{Re}(\chi^*(\tau)y(\tau)) + 2 \operatorname{Re}(\chi(\tau)\chi(\tau)y^*(2\tau)) \tag{9c}$$

$$y(\tau) = W \int_{-\infty}^{+\infty} |\chi(t)|^2 |\chi(t+\tau)|^2 \, dt. \tag{9d}$$

Equation (8) is thus a general expression for the variance of the statistic $\hat{g}$ as a function of the incident light's lineshape, degree of coherence and intensity level. A special but very important case is that when the light has a lorentzian spectrum, that is,

$$\chi(\tau) = \exp(-W|\tau| + i\omega_0\tau).$$

In this case,

$$a(0) = \tfrac{1}{2} \tag{10a}$$

$$u(\tau) = -2 + 8\exp(-2x) - \exp(-4x) \tag{10b}$$

$$v(\tau) = -4\exp(-2x) - (\tfrac{3}{2} + 2x)\exp(-4x), \tag{10c}$$

where $x = |\tau W|$.

We note that when only one detector is used, $\Gamma_{12} = 1$, the combination of (8) and (10*a*, *b*, *c*) reproduces the results previously published by Jakeman *et al* (1971a). Our result (8) is a more general expression in that it considers measurements by two detectors instead of one and in that it is correct for all lineshapes. However, it is a special case in that we assume $T_d \ll \tau_c$.

5. The estimation problem

The magnitude of the spatial part of the degree of coherence $|\Gamma_{12}|$ is assumed to depend on a parameter θ. In this section we proceed to find an estimate for θ, say $\hat{\theta}$, given that the statistic $\hat{g}$ is observed. Since, for large TW, $\hat{g}$ is approximately an unbiased estimator of g with a small variance proportional to $(TW)^{-1}$, it seems reasonable, and simple, to choose $\hat{\theta}$ as the value of θ that satisfies

$$\hat{g} = g(\hat{\theta}). \tag{11}$$

In order to find the variance of $\hat{\theta}$, we expand $g(\hat{\theta})$ around the correct unknown value θ, thus

$$\hat{g} = g(\hat{\theta}) = g(\theta) + \frac{\partial g(\theta)}{\partial \theta}\delta\hat{\theta},$$

where $\delta\hat{\theta} = \hat{\theta} - \theta$. This gives

$$\delta\hat{\theta} \simeq |\hat{g} - g(\theta)| \left(\frac{\partial g(\theta)}{\partial \theta}\right)^{-1}.$$

This shows that $\hat{\theta}$ is approximately an unbiased estimator of θ. It also shows that

$$\operatorname{var}(\hat{\theta}) \simeq \operatorname{var}(g)\left|\frac{\partial g}{\partial \theta}\right|^{-2}. \tag{12}$$

The estimation based on (11) is obviously not the best that can be done. If the probability distribution of g is available a maximum-likelihood estimator can be found. This is a problem we do not attempt. It is interesting to note that if g is gaussian (this is approximately the case when TW is very large) then the Kramér–Rao lower bound on the variance of unbiased estimators of θ takes, in the first approximation, the same value given by (12) (see eg Mood and Graybill 1963).

Thus, with the combination of (8) and (12), we have found an expression for the accuracy in estimating any parameter of the intensity correlation function $g(\theta)$. In the following section we give two examples of applications of the above results.

6. Examples

6.1. *Estimation of the spatial coherence length of a homogeneous optical field, with lorentzian spectrum*

In this example $\Gamma_{12} = \exp(-|r_1 - r_2|/r_c)$, and r_c, the effective spatial coherence length, is to be estimated. By simple substitution of Γ_{12} in (8) and use of (10*a*, *b*, *c*), (12) yields an expression for the required estimation error. We have found numerically that the choice of $\tau = 0$ gives the least error. In this case

$$\begin{aligned} e_{\hat{r}_c} &= \operatorname{var}^{1/2}(r_c)/r_c \\ &= (TW)^{-1/2}\Bigg(\tfrac{1}{2}\{5\exp(4\xi)+10\exp(2\xi)-11\} \\ &\quad +\frac{2}{\bar{n}_c}\{\exp(4\xi)+2\exp(2\xi)-1\}+\frac{1}{\bar{n}_c\bar{n}}\{\exp(4\xi)+\exp(2\xi)\}\Bigg)^{1/2}\frac{1}{2\xi} \end{aligned}$$

where $\xi = |r_1 - r_2|/r_c$. This error is plotted (figure 1) for several values of $\bar{n}$ and $\bar{n}_c$. It is interesting to see that the error reaches a minimum when $|r_1 - r_2| \simeq \frac{1}{2}r_c$.

6.2. *Estimation of the radius of an incoherent circular object by measurements at an aperture far from the object plane*

We assume an incoherent circular object of radius b and radiance function $B(u)$ at a point u of the object plane. We also assume the radiated light to have a lorentzian spectrum. Light radiated from this object propagates deterministically, and produces under conditions of paraxial rays an aperture field with coherence (see eg Mandel and Wolf 1965)

$$\Gamma_{12} = 2J_1(y)/y,$$

where $y = 2\pi b|r_1 - r_2|/\lambda R$, where R is the distance between the object and aperture planes, λ is the light wavelength and J_1 is the Bessel function of order one. Again by

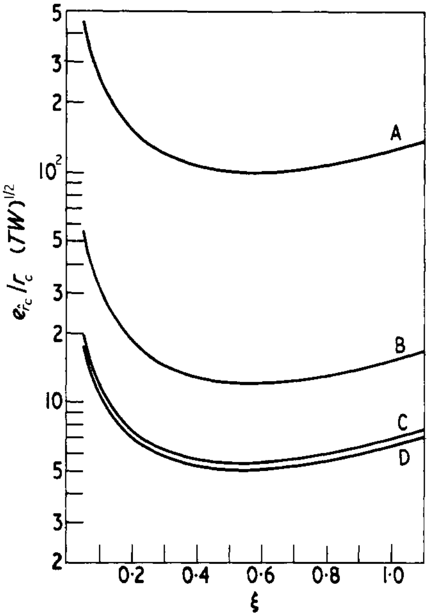

Figure 1. The error in estimating the spatial coherence length r_c as a function of the normalized distance between the photodetectors $\xi = |r_1 - r_2|/r_c$, for several values of the average counting rates $\bar{n}$ and $\bar{n}_c = \bar{n}(\tau_c/T_d)$. A, $\bar{n} = 0{\cdot}01$, $\bar{n}_c = 0{\cdot}1$; B, $\bar{n} = 0{\cdot}1$, $\bar{n}_c = 1{\cdot}0$; C, $\bar{n} = 1{\cdot}0$, $\bar{n}_c = 10$; D, $\bar{n} = \infty$, $\bar{n}_c = \infty$.

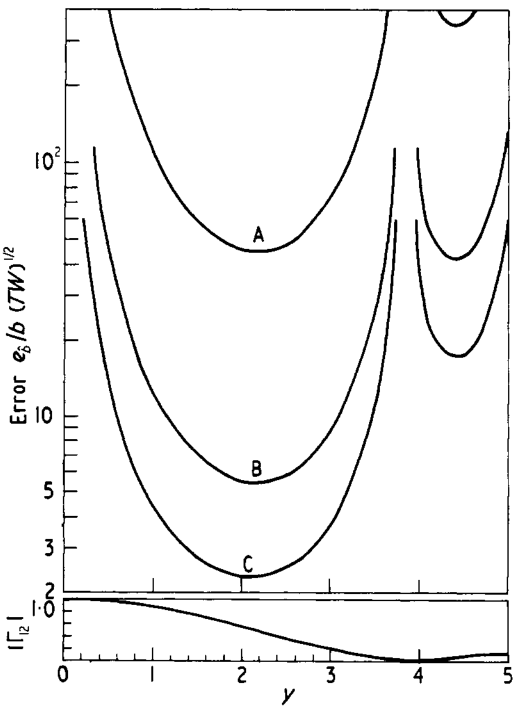

Figure 2. The error in estimating the radius of an incoherent object b as a function of $y = 2\pi b|r_1 - r_2|/\lambda R$ for several counting rates $\bar{n}$ and $\bar{n}_c$: A, $\bar{n} = 0{\cdot}01$, $\bar{n}_c = 0{\cdot}1$; B, $\bar{n} = 0{\cdot}1$, $\bar{n}_c = 1{\cdot}0$; C, $\bar{n} = \infty$, $\bar{n}_c = \infty$. Also shown a plot of the degree of coherence $|\Gamma_{12}|$ against y.

substituting in (8), (10) and (12) with $\tau = 0$, we get

$$e_{\hat{b}} = \mathrm{var}^{1/2}(b)/b$$

$$= (TW)^{-1/2}\left[\frac{5}{2}\left(\frac{2J_1(y)}{y}\right)^{-2} + 5 - \frac{11}{2}\left(\frac{2J_1(y)}{y}\right)^{2} + \frac{2}{\bar{n}_c}\left\{\left(\frac{2J_1(y)}{y}\right)^{-2} + 2 - \left(\frac{2J_1(y)}{y}\right)^{2}\right\} + \frac{1}{\bar{n}_c\bar{n}}\left\{1 + \left(\frac{2J_1(y)}{y}\right)^{-2}\right\}\right]^{1/2}\frac{1}{4J_2(y)}.$$

The variation of the above error with y is plotted in figure 2 which shows the values of y (ie the separations $|r_1 - r_2|$) at which the error is minimum.

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